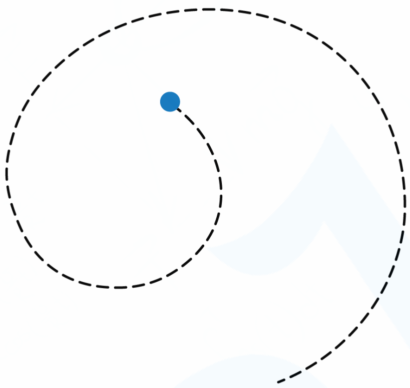
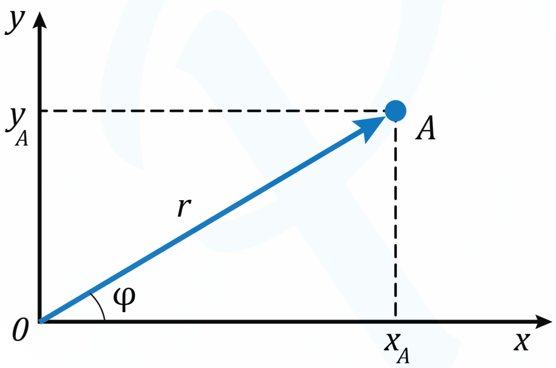
Polar coordinate system

- Answers and detailed solutions to all problems are provided in iOS/Android "**PhysOlymp**" app
- With any suggestions please write to **feedback@physolymp.com**

Selection of fit for purpose coordinate system can significantly simplify solution of the problem. For the cases when trajectory of the particle is close to circular or spiral like contour in a plane, it is usually convenient to use polar coordinates



Polar coordinate system is characterized by a distance r from the reference point O and angle φ from some reference direction (Ox)



Location of some specific point A in Cartesian system in terms of known polar coordinates can be expressed as

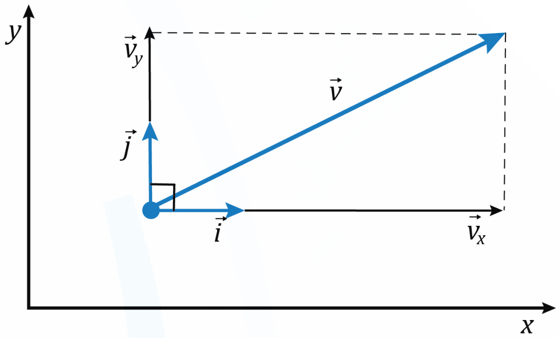
$$x_A = r\cos\varphi$$

$$y_A = r\sin\varphi$$

Components of velocity $\vec{v}$ in Cartesian coordinates are described with two unit vectors $\vec{i}$ and $\vec{j}$, which are parallel to x and y axises respectively, with a direction alongside of increasing values of co-ordinates x and y

$$\vec{v}_x = \frac{dx}{dt}\vec{i}$$

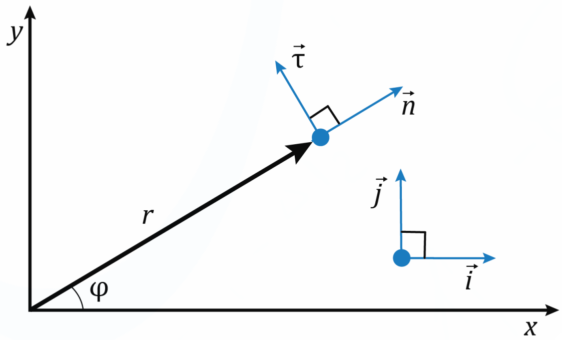
$$\vec{v}_y = \frac{dy}{dt}\vec{j}$$



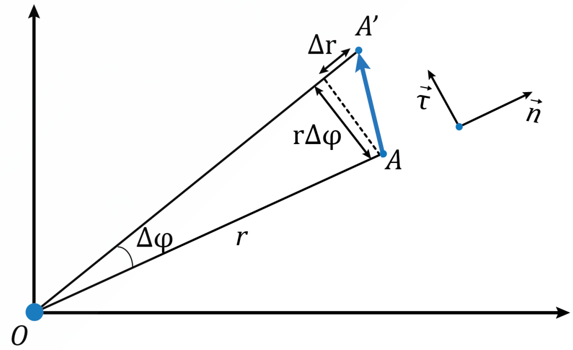
Total velocity in Cartesian coordinate system is a vector sum of its components:

$$\vec{v} = \vec{v}_x + \vec{v}_y = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

Similarly, directions in polar coordinate system are characterized with two unit vectors: one with a normal $\vec{n}$ outward direction and another one perpendicular to vector $\vec{n}$ with a counterclock-wise direction $\vec{\tau}$



By definition of velocity, it is a displacement in unit of time. Let's consider a motion in plane from the point A to the position A'



Normal component of velocity v_n can be defined as displacement Δr from the origin of coordinates in a small time interval Δt

$$v_n = \frac{\Delta r}{\Delta t}$$

With accounting to direction described by unit vector $\vec{n}$ this can be rewritten as

$$\vec{v}_n = \frac{\Delta r}{\Delta t}\vec{n} = \frac{dr}{dt}\vec{n} = \dot{r}\vec{n} \quad (1)$$

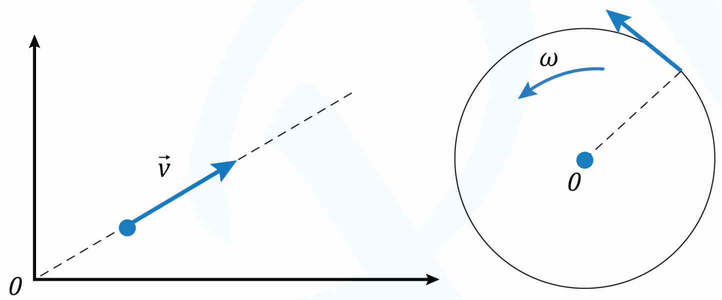
Similarly tangential component of velocity can be defined as displacement $r\Delta\varphi$ obtained by rotating at small angle $\Delta\varphi$ during time interval Δt

$$v_\tau = \frac{r\Delta\varphi}{\Delta t}$$

Direction of tangential component of velocity coincides with unit vector τ. Then,

$$\vec{v}_\tau = \frac{r\Delta\varphi}{\Delta t}\vec{\tau} = r\frac{d\varphi}{dt}\vec{\tau} = r\dot{\varphi}\vec{\tau} \quad (2)$$

Equations (1) and (2) can be easily comprehended by considering limiting cases, with trajectory of the moving particle being either a straight line or a circle



For a motion along a straight line equation (1) becomes a regular definition of velocity as a measure of change of coordinate r per unit of time

$$v_n = \frac{dr}{dt} = \dot{r}$$

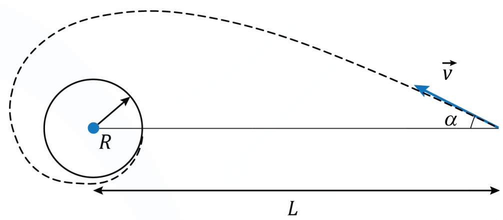
For another limiting case of the motion around a fixed center with constant radius r, velocity of the object can be defined by its angular velocity $\omega = \dot{\varphi}$ as

$$v_\tau = \omega r = \dot{\varphi} r$$

which corresponds to a general form of tangential component of velocity in polar coordinate system given by equation (2)

Example 1

A moth flies towards the lamp with constant velocity, which always has some angle α with radial direction. How many turns N will make the moth before hitting the lamp? Initial distance between the moth and the lamp is L, radius of the lamp is R



Spiral like trajectories are more convenient to describe by using polar coordinates. If the moth is moving with velocity v, then its components of velocity in the normal and tangential directions of the polar coordinate system can be defined as

$$\frac{dr}{dt} = -v\cos\alpha$$

$$r\frac{d\varphi}{dt} = v\sin\alpha$$

Dividing those equations gives

$$\frac{dr}{rd\varphi} = -\frac{1}{\tan\alpha}$$

Separating terms allow to perform a trivial integration

$$\int_L^R \frac{dr}{r} = -\frac{1}{\tan\alpha}\int_0^{2\pi N} d\varphi$$

Finally,

$$\boxed{N = \left[\tan\alpha \ln\left(\frac{L}{R}\right)\right]}$$

where was taken into account that number of revolutions is integer number

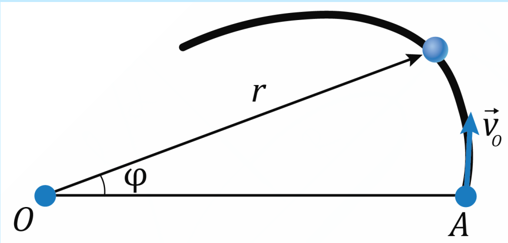
Remark: a trajectory with a constant angle at the tangent is called logarithmic spiral. It frequently appears in nature, such as for the shape of nautilus shell, extratropical cyclone, or the arms of spiral galaxies

Problem 1

A small bead can move along a fixed rigid wire, which has a shape defined in polar coordinate system as

$$r = R|\cos\varphi|$$

where $R = 1.0\ m$, φ is angle counted from initial position in the point A as shown at the picture



Estimate angular velocity ω of the bead, after $t = 0.1\ s$, if its initial velocity is $v_0 = 1.0\ m/s$. Friction, gravity or any other external forces can be neglected. Angular velocity is defined as

$$\omega = \dot{\varphi} = \frac{d\varphi}{dt}$$

Acceleration in polar coordinate system

Derivations for normal and tangential components of the acceleration in polar coordinates are a little bit more tricky. Let's start from the definition of acceleration a as a change of velocity Δv per time interval Δt

$$a = \frac{\Delta v}{\Delta t}$$

As acceleration and velocity are vectors, then components of acceleration along normal and tangential direction can be described as

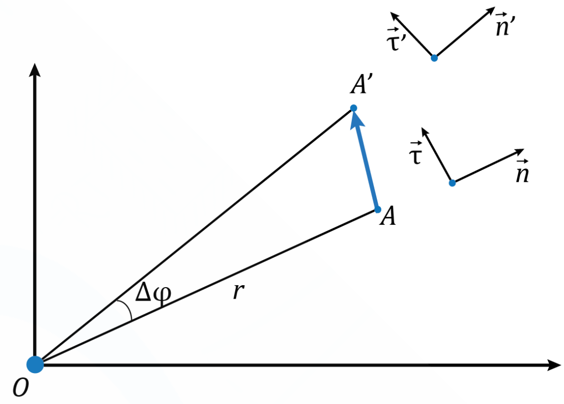
$$\vec{a}_n = \frac{d\vec{v}_n}{dt}; \qquad \vec{a}_\tau = \frac{d\vec{v}_\tau}{dt}$$

Using derived earlier equations of components of velocity in polar coordinate system, and chain rule for taking derivatives results in following

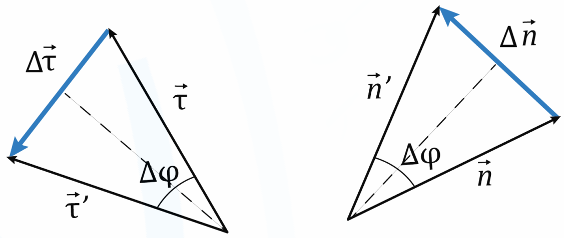
$$\vec{a}_n = \frac{d\vec{v}_n}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\right)\cdot\vec{n} + \frac{dr}{dt}\cdot\frac{d\vec{n}}{dt} \tag{1}$$

$$\vec{a}_\tau = \frac{d\vec{v}_\tau}{dt} = \frac{d}{dt}\left(r\frac{d\varphi}{dt}\right)\cdot\vec{\tau} + r\frac{d\varphi}{dt}\cdot\frac{d\vec{\tau}}{dt} \tag{2}$$

After displacement from initial position A to the point A', absolute magnitude of unit vectors $|\vec{n}| = \left|\vec{n}'\right| = 1$ and $|\vec{\tau}| = \left|\vec{\tau}'\right| = 1$ is constant, but they change direction from $\vec{n}$ to $\vec{n}'$ and from $\vec{\tau}$ to $\vec{\tau}'$ as shown at the picture below



Rate of change $d\vec{n}/dt$ and $d\vec{\tau}/dt$ can be estimated by examining small isosceles triangles formed by pairs of vectors $\vec{\tau}$ and $\vec{\tau}'$ as well as $\vec{n}$ and $\vec{n}'$:



From isosceles triangle can be obtained

$$\frac{d\vec{n}}{dt} = \frac{\Delta n}{dt}\cdot\vec{\tau} = \frac{2\sin\frac{\Delta\varphi}{2}|\vec{n}|}{dt}\cdot\vec{\tau}$$

For a small angle $\Delta\varphi \ll 1$ can be used approximation

$$\sin\frac{\Delta\varphi}{2} \approx \frac{\Delta\varphi}{2}$$

Then,

$$\frac{d\vec{n}}{dt} = \frac{\Delta\varphi}{dt}\cdot\vec{\tau} = \frac{d\varphi}{dt}\cdot\vec{\tau} \tag{3}$$

Similar for tangential unit vector

$$\frac{d\vec{\tau}}{dt} = -\frac{\Delta\tau}{dt}\cdot\vec{n} = -\frac{2\sin\frac{\Delta\varphi}{2}|\vec{\tau}|}{dt}\cdot\vec{n} = -\frac{d\varphi}{dt}\cdot\vec{n} \tag{4}$$

Negative sign is accounting for inward direction, which is opposite to the direction of unit vector $\vec{n}$

Expanding terms in equations (1) and (2) with using expressions (3) and (4) yields

$$\vec{a}_n = \ddot{r}\vec{n} + \dot{r}\frac{d\varphi}{dt}\vec{\tau}$$

$$\vec{a}_\tau = \dot{r}\frac{d\varphi}{dt}\vec{\tau} + r\ddot{\varphi}\vec{\tau} - r\frac{d\varphi}{dt}\cdot\frac{d\varphi}{dt}\vec{n}$$

Combining both equations results in

$$\vec{a} = \vec{a}_n + \vec{a}_\tau = (\ddot{r} - r\dot{\varphi}^2)\vec{n} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\vec{\tau}$$

Or

$$\boxed{|a_n| = \ddot{r} - r\dot{\varphi}^2}$$

which is equivalent to the effective acceleration for the case, when moving with inertial rotating frame. In that frame a particle will move along a straight line in the radial direction, so its acceleration $\ddot{r}$ should be just corrected for centrifugal acceleration $r\dot{\varphi}^2$

Similarly for a tangential component

$$|a_\tau| = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = \frac{1}{r}\frac{d(r^2\dot{\varphi})}{dt}$$

In case of central forces

$$|a_\tau| = \frac{F_\tau}{m} = 0$$

which corresponds to

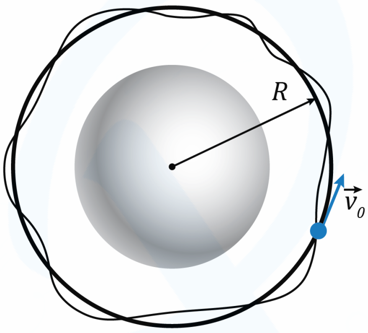
$$r^2\dot{\varphi} = const$$

This is a regular expression for conservation of angular momentum, in centrally symmetrical fields

Example 2

A satellite is moving around the Earth at a circular orbit with radius R. After collisions with small debris the satellite starts oscillating in the radial direction. Find period of those small oscillations T of the satellite, assuming that its trajectory is still close to circular

Let's describe motion of the satellite, when it is at the distance r from the center of the Earth



As trajectory of the body is close to circular, we can try to work with polar coordinate system. Using result of derivation for normal component of acceleration in polar coordinate system, Newton's law for the satellite can be written in radial direction as

$$m(\ddot{r} - \dot{\varphi}^2 r) = -\frac{GMm}{r^2} \quad (1)$$

Negative sign in the right side of the equation accounts for the fact that gravity force is attractive, with direction to the center of the Earth, which is opposite to the regular notations of polar coordinate system. As there is no tangential forces applied, then angular momentum of the satellite is conserved:

$$r^2\dot{\varphi} = const = v_0 R \quad (2)$$

where v_0 is initial velocity of the satellite, when it was rotating at the circular orbit with radius R. Balance of forces for circular motion can be described as

$$\frac{mv_0^2}{R} = \frac{GMm}{R^2} \quad (3)$$

After combining all equations together we will get following relation

$$\ddot{r} - \frac{GMR}{r^3} + \frac{GMm}{r^2} = 0 \quad (4)$$

if $r = R + x$, where $x \ll R$, then equation (4) can be modified as

$$\ddot{x} - \frac{GM}{R^2\left(1+\frac{x}{R}\right)^3} + \frac{GMm}{R^2\left(1+\frac{x}{R}\right)^2} = 0 \quad (5)$$

For small parameter $z \ll 1$ can be used approximation

$$(1+z)^n \approx 1 + nz$$

So, equations (5) can be simplified to

$$\ddot{x} - \frac{GM}{R^2}\left(1 - 3\frac{x}{R}\right) + \frac{GMm}{R^2}\left(1 - 2\frac{x}{R}\right) = 0$$

Or,

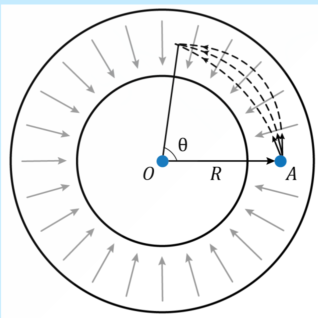
$$\ddot{x} + \frac{GM}{R^3}x = 0$$

This is a classical equation for a simple harmonic oscillator with a period of oscillations

$$T = 2\pi\sqrt{\frac{R^3}{GM}}$$

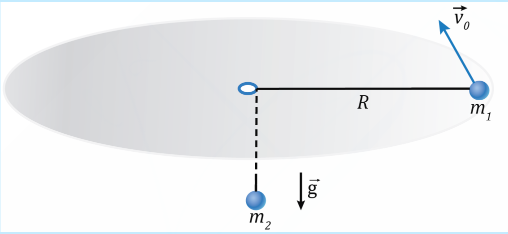
Problem 2

A beam of electrons is emitted at the point A, which is located between the plates of the long cylindrical capacitor. Electrons are emitted not at the same direction, but rather have some small spread angle at the injection point, which is causing the beam to expand. However, due to electric field from the capacitor the beam focuses again at some point characterized with angle θ as shown at the picture. Find angle θ, if known that electric field between the plates of the capacitor is radially symmetrical and proportional to inverse of the distance from the axis of symmetry



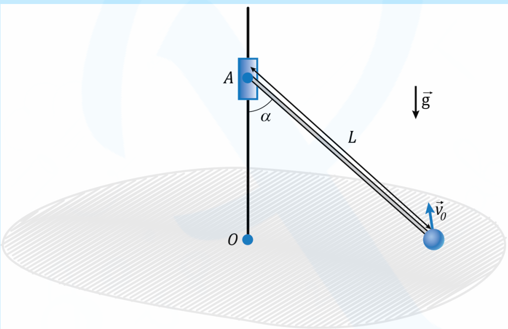
Problem 3

Two small objects with masses $m_1 = 0.1\ kg$ and $m_2 = 0.2\ kg$ are tied with a lightweight, inextensible rope of the length $L = 1.0\ m$. Initially, the weight with a mass m_1 is moving around a circular trajectory with radius $L/2$ at the flat frictionless table. The second body is suspended on the rope, which goes through a small hole in the table. Find period of small oscillations T of the second body in the vertical direction. For calculations use $g = 9.8\ m/s^2$



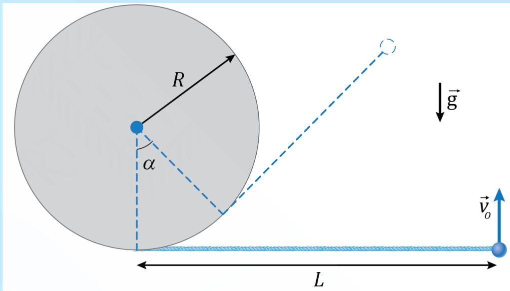
Problem 4

A coupling can move along the vertical axis OA without friction. One end of the weightless rigid rod with a length $L = 1.0\ m$ is fastened to the joint A, while to the other end of the rod is attached a heavy ball of a small size, which can move on the flat smooth surface. Initially, the system is at rest and the ball is pushed in such a way, that it has initial velocity $v_0 = 1.0\ m/s$ in the direction along horizontal surface and perpendicular to the rod. Find acceleration of the coupling a_0 in the beginning of motion of this system. For calculations assume that mass of the ball is equal to mass of the coupling; initial angle between the rod and the vertical axis is $\alpha_0 = 60°$, while acceleration due to gravity is $g = 9.8\ m/s^2$



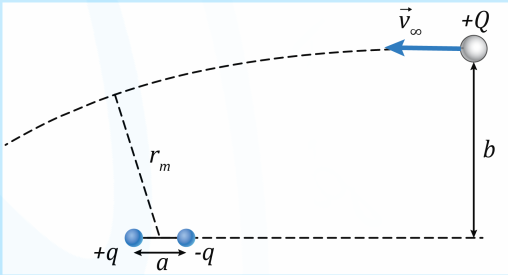
Problem 5

A small object is attached to the weightless non-stretchable string, which can wrap around a smooth cylinder with radius $R = 1.0\ m$. Initially, a linear piece of the string is horizontal with a length $L = \pi R$, while the bob is pushed with velocity v_0 in the vertical direction. Find value of v_0, if known that tension force of the string became zero after wrapping on the cylinder by angle $\alpha = 60°$. For calculations use gravity $g = 9.8\ m/s^2$



Problem 6

A small particle with a mass m and charge $+Q$ is passing by a fixed dipole with charges $\pm q$ separated by tiny interval a. Find minimal distance r_m between the dipole and moving particle, if its impact parameter is $b = 1 \cdot 10^{-3} m$, $(b \gg a)$.

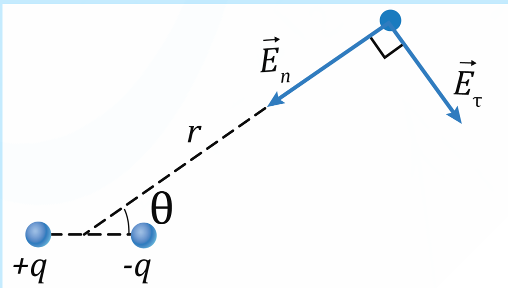


Assume that friction, gravity or any other forces except interaction between the charged particle and dipole can be neglected. For numerical calculations use dimensionless parameter γ:

$$\gamma = \frac{kQqa}{mv_\infty^2 b^2} = \frac{1}{3}$$

where v_∞ is initial velocity of the particle at the large distance from the dipole

Hint:



Normal E_n and tangential E_τ components of the electrical field of the dipole can be described as

$$E_n = \frac{2kqa\cos\theta}{r^3}$$

$$E_\tau = \frac{kqa\sin\theta}{r^3}$$

where r is distance from the dipole and θ is angle as shown at the picture above. Potential φ from the dipole electrical field is

$$\varphi = -\frac{kqa\cos\theta}{r^2}$$